Research Note on New Similarity Reductions of a Variable-Coefficient Korteweg-de Vries Equation

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For a variable-coefficient Korteweg-de Vries equation we obtain 4 new similarity reductions to the Painlevé type equations or the Weierstrass elliptic function equation.

In this note, we report the application of the Clarkson-Kruskal direct method [1], which is powerful for the constant-coefficient nonlinear evolution equations, to a variable-coefficient Korteweg-de Vries equation [2],

$$u_t + f(t)\, u\, u_x + g(t)\, u_{xxx} = 0, \tag{1}$$

where $f(t) \neq 0$ and $g(t) \neq 0$.

Firstly, it is sufficient to seek a similarity reduction of (1) in the form

$$u(x,t) = a(x,t) + b(x,t)\, w[z(x,t)], \tag{2}$$

where the functions $a(x,t)$, $b(x,t)$, $z(x,t)$ and $w(z)$ are to be determined, rather than the more general form of $u(x,t) = U\{x, t, w[z(x,t)]\}$ (proof ignored). Also, we will consider the interesting case of $z_x \neq 0$.

After the substitution of (2), we impose the condition that (1) be an ordinary differential equation for $w(z)$, i.e., demand the ratios of the coefficients of different derivatives and powers of $w(z)$ to be functions of z only. Computerized symbolic computation thus leads to a set of conditions for $a(x,t)$, $b(x,t)$ and $z(x,t)$, based on which we find the following format of similarity reductions:

$$u(x,t) = -\frac{\theta'(t)\, x + \phi'(t)}{f(t)\, \theta(t)} + \frac{g(t)}{f(t)}\, z_x^2(x,t)\, w[z(x,t)], \tag{3}$$

with $$z(x,t) = \theta(t)\, x + \phi(t), \tag{4}$$

where the functions $\theta(t) \neq 0$, $\phi(t)$, $w(z)$, as well as the constraints on $f(t)$ and $g(t)$, are given below case by case:

Similarity reduction I:

$$\theta(t) = k, \tag{5}$$

$$\phi(t) = \begin{cases} C\, k^6 \int^t f(t) \int^s \frac{g^2(s)}{f(s)}\, \mathrm{d}s\, \mathrm{d}t & \text{when} \quad C \neq 0, \\ M \int f(t)\, \mathrm{d}t & \text{otherwise}, \end{cases} \tag{6}$$

where C, $k \neq 0$ and M are arbitrary constants, while $w(z)$ satisfies the ordinary differential equation

$$w''' + w\, w' + C = 0, \tag{7}$$

which is equivalent to either the first Painlevé equation as $C \neq 0$ or the Weierstrass elliptic function equation as $C = 0$.

The constraint on $f(t)$ and $g(t)$ is

$$\Lambda\left[\frac{g(t)}{f(t)}\right] = 0, \quad \text{where} \quad \Lambda \equiv \frac{\mathrm{d}}{\mathrm{d}t} \ln. \tag{8}$$

Similarity reduction II:

$$\theta(t) = E\, \frac{f(t)}{g(t)}, \tag{9}$$

while $\phi(t)$ satisfies

$$\frac{f(t)}{g^2(t)\, \theta^5(t)}\left[\frac{f'(t)\, \phi'(t)}{f^2(t)\, \theta(t)} + \frac{2\theta'(t)\, \phi'(t)}{f(t)\, \theta^2(t)} - \frac{\phi''(t)}{f(t)\, \theta(t)}\right] = B\, \phi(t) + D, \tag{10}$$

where D and $E \neq 0$ are arbitrary constants, while B is a constant given by (13) below, and $\theta(t)$ by (9). $w(z)$ must satisfy

$$w''' + w\, w' + B\, z + D = 0, \tag{11}$$

which is equivalent to the first Painlevé equation when $B = 0$, or else, not of the Painlevé type.

The constraints on $f(t)$ and $g(t)$ are

$$\Lambda\left[\frac{g(t)}{f(t)}\right] \neq 0, \tag{12}$$

$$\frac{f(t)}{g^2(t)\, \theta^6(t)}\left\{\frac{f'(t)\, \theta'(t)}{f^2(t)\, \theta(t)} + \frac{2[\theta'(t)]^2}{f(t)\, \theta^2(t)} - \frac{\theta''(t)}{f(t)\, \theta(t)}\right\} = B = \text{constant}, \tag{13}$$

with $\theta(t)$ given by (9).

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Similarity reduction III:

$$\theta(t) = -[3A \int g(t)\,\mathrm{d}t]^{-1/3}, \tag{14}$$

$$\phi(t) = \frac{H}{2A^2} - [3A \int g(t)\,\mathrm{d}t]^{-1/3}, \tag{15}$$

where $A \neq 0$ and H are arbitrary constants, while $w(z)$ satisfies

$$w''' + w\,w' + A\,w - 2A^2\,z + H = 0, \tag{16}$$

the solutions of which are known to be related through a one-to-one transformation to those of the second Painlevé equation.

The constraint on $f(t)$ and $g(t)$ is

$$\Lambda\left[\frac{g(t)}{f(t)}\right] = 0, \quad \text{where} \quad \Lambda \equiv \frac{\mathrm{d}}{\mathrm{d}t}\ln. \tag{17}$$

Similarity reduction IV:

$$\theta(t) = -\frac{f(t)}{g(t)}\left[3N \int \frac{f^3(t)}{g^2(t)}\,\mathrm{d}t\right]^{-1/3}, \tag{18}$$

$$\phi(t)\begin{cases} \text{satisfies} \quad \dfrac{f'(t)\,\phi'(t)}{f^2(t)\,\theta(t)} + \dfrac{2\theta'(t)\,\phi'(t)}{f(t)\,\theta^2(t)} - \dfrac{\phi''(t)}{f(t)\,\theta(t)} = P\,\dfrac{g^2(t)}{f(t)}\,\theta^5(t) & \text{if} \quad P \neq 0 \;\&\; R = 0, \\ = \theta(t) & \text{if} \quad P = 0 \;\&\; R = 0, \\ = \theta(t) - P/R & \text{otherwise}, \end{cases} \tag{19}$$

where $N \neq 0$ and P are arbitrary constants, while R is a constant given by (22) below, and $\theta(t)$ by (18). $w(z)$ satisfies

$$w''' + w\,w' + N\,w + R\,z + P = 0, \tag{20}$$

which is a form of the second Painlevé equation only when $R + 2N^2 = 0$.

The constraints on $f(t)$ and $g(t)$ are

$$\Lambda\left[\frac{g(t)}{f(t)}\right] \neq 0, \tag{21}$$

$$\frac{R}{N^2} = -2 + 9\,\frac{g^4(t)}{f^5(t)}\left[\int \frac{f^3(t)}{g^2(t)}\,\mathrm{d}t\right]^2 \cdot \left\{\frac{3[f'(t)]^2}{f^3(t)} - \frac{3f'(t)\,g'(t)}{f^2(t)\,g(t)} - \frac{f''(t)}{f^2(t)} + \frac{g''(t)}{f(t)\,g(t)}\right\}, \tag{22}$$

with $\theta(t)$ given by (18).

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[1] P. Clarkson and M. Kruskal, J. Math. Phys. **30,** 2201, 1989.
[2] N. Joshi, Phys. Lett. **A 125,** 456, 1987.